Design Methodology for Robust Stabilizing Controllers

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This paper considers the problem of designing control laws for linear systems with time-varying uncertainty. Lyapunov stability theory is used to develop a numerical method of finding a control law that asymptotically stabilizes such systems. This control is robust in the sense that it guarantees asymptotic stability regardless of the disturbance. The results are applied to several aircraft examples.

Introduction

CONSIDER the uncertain linear system

$$\dot{x}(t) = [A_0 + \Delta A(r(t))]x(t) + [B_0 + \Delta B(s(t))]u(t) \quad (1)$$

where $x(t) \in R^n$ is the *state*, $u(t) \in R^m$ is the *control*, and A_0 and B_0 are the nominal system matrices. We assume the nominal system is controllable and the matrix B_0 has full rank.

Uncertainty in the system is modeled by the functions $r(\cdot)$ and $s(\cdot)$, with $r(t) \in R^p$ and $s(t) \in R^\ell$. The only information available about the uncertainty is that the uncertain elements $r(t)$ and $s(t)$ belong to known compact sets; $r(t) \in \mathscr{R}$ and $s(t) \in \mathscr{S}$. The set of functions satisfying $r(t) \in \mathscr{R}$ will be denoted by $M(\mathscr{R})$; $M(\mathscr{S})$ is similarly defined. The most common type of constraint sets are of the form

$$\mathscr{R} = \{r(t) \in R^p\colon \underline{r}_i \leq r_i(t) \leq \bar{r}_i,\ i = 1, 2, \ldots, p\}$$

$$\mathscr{S} = \{s(t) \in R^\ell;\ \underline{s}_i \leq s_i(t) \leq \bar{s}_i,\ i = 1, 2, \ldots, \ell\}$$

In this case, only the upper and lower bounds $\bar{r}_i$, $\bar{s}_i$, $\underline{r}_i$, and $\underline{s}_i$ are known.

We now state our problem.

Problem. Determine a linear feedback control law $u(t) = Kx(t)$, such that $x(t) \to 0$ for all $r(\cdot) \in M(\mathscr{R})$, $s(\cdot) \in M(\mathscr{S})$, and for all initial conditions $x_0 \in R^n$. Here K is a constant $m \times n$ matrix.

A control $u(\cdot)$, which accomplishes this objective, will be called a *robust linear stabilizing control.* Determining a robust linear stabilizing control is equivalent to determining the matrix K.

In addition to the problem where the uncertainty is time-varying, we also treat as a special case problems where the uncertainty is unknown but constant, i.e., $r(t) = \bar{r}$, $s(t) = \bar{s}$, and $\bar{r} \in \mathscr{R}$, $\bar{s} \in \mathscr{S}$. Our examples show that if, for given $\mathscr{R}$ and $\mathscr{S}$, a control law is a robust control for constant uncertainty, it may not be a robust control for time-varying uncertainty with the same bounds. To guarantee stability when the uncertainty is time-varying requires a "larger" gain matrix K.

In the next section, a method for obtaining a robust stabilizing control law is presented. To discuss the effectiveness of this control, we need to consider two classes of systems—those that are matched and those that are not.

Definition. The system (1) is *matched* if there exist continuous matrix functions $D(\cdot)$ and $E(\cdot)$, such that $I + \frac{1}{2}[E(s) + E'(s)]$ is positive definite for all $s \in \mathscr{S}$ and

$$\Delta A(r) = B_0 D(r), \qquad \Delta B(s) = B_0 E(s)$$

for all $r \in \mathscr{R}$ and $s \in \mathscr{S}$. If Eq. (1) does not satisfy these conditions, the system is said to be a *mismatched* system.

The robust stabilization problem has received considerable attention in recent years. Nonlinear feedback control laws that guarantee stability were obtained by Leitmann[1,2] for systems satisfying the matching assumption. Linear robust stabilizing control laws were presented by Thorp and Barmish[3] for both matched systems and systems satisfying a less stringent generalized matching condition. However, the determination of these control laws requires transformation of the system to a canonical form and also involves complicated calculations, which often lead to high gains. The same difficulties—complicated calculations and high gains—arise in designing controllers for matched systems via the technique of Barmish et al.[4] or in the stability counterpart of the tracking result of Schmitendorf and Barmish.[5] More recently, Peterson[6] has devised a scheme for computing stabilizing controllers, which involves solving a sequence of Riccati-type equations parameterized by a scalar. Here we give a method for obtaining purely linear feedback laws that are relatively simple to determine via numerical computation. Furthermore, the required gains are smaller than those obtained in Refs. 3–5. The design method has been applied successfully to both matched and mismatched systems.

Controller Design Method

First, we give a procedure for constructing a particular feedback control law and then give conditions under which it is a robust linear stabilizing control. Our procedure is based on the construction given in Schmitendorf and Barmish[5] for tracking problems with constant uncertainty.

1) Consider the nominal system

$$\dot{x}(t) = A_0 x(t) + B_0 u(t) \quad (2)$$

and choose $u(t) = K_0 x(t)$ to stabilize this system, i.e., choose K_0 so that all eigenvalues of $\tilde{A}_0 = A_0 + B_0 K_0$ have negative real parts.

This stabilization of the nominal system can be accomplished by eigenvalue placement or by using linear-quadratic optimal control theory.

2) Choose any positive-definite symmetric matrix Q and solve the Lyapunov equation

$$\tilde{A}_0' P + P \tilde{A}_0 = -Q \quad (3)$$

for the positive-definite symmetric matrix P. Usually Q is taken as the identity matrix.

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3) Choose a scalar $\gamma > 0$ such that $L(r, s)$ is negative definite for all $r \in \mathscr{R}$, $s \in \mathscr{S}$, where

$$L(r, s) \triangleq -Q + P\Delta A(r) + \Delta A'(r)P + P\Delta B(s)K_0 + K_0'\Delta B'(s)P - 2\gamma PB_0B_0'P - \gamma[P\Delta B(s)B_0'P + PB_0\Delta B'(s)P]$$

4) Let

$$u(t) = (K_0 - \gamma B_0'P)x(t) \tag{4}$$

Our main result is: *A control obtained via steps* 1–4 *is a robust stabilizing control.*

This result is established by showing that the Lyapunov function $V = x'Px$ [P given by Eq. (3)] has negative trajectory derivative. With the control given by Eq. (4), the closed-loop system satisfies

$$\dot{x} = [\tilde{A}_0 + \Delta A(r) + \Delta B(s)K_0 - \gamma B_0B_0'P - \gamma\Delta B(s)B_0'P]x$$

Thus,

$$\begin{aligned}\dot{V} &= 2x'P[\tilde{A}_0 + \Delta A(r) + \Delta B(s)K_0 - \gamma B_0B_0'P - \gamma\Delta B(s)B_0'P]x \\ &= x'P[-Q + P\Delta A(r) + \Delta A'(r)P + P\Delta B(s)K_0 \\ &\quad + K_0'\Delta B'(s)P - 2\gamma B_0B_0'P - \gamma(P\Delta B(s)B_0'P \\ &\quad + PB_0\Delta B'(s)P)]x \\ &= x'L(r, s)x\end{aligned}$$

If γ is chosen as in step 3, this derivative is negative for all $x \neq 0$ and, from standard Lyapunov theory, the origin is asymptotically stable.

The main question that arises in determining the control from Eq. (4) is whether there exists a γ satisfying the condition in step 3. If the system is matched, there always exists a γ^* such that for $\gamma \geq \gamma^*$, the negative-definite condition on L in step 3 is satisfied.[5] This result can be established using an argument identical to that in Ref. 5. For mismatched systems, the existence of an appropriate γ has not been proven and, in fact, for some problems, it may not exist. However, in almost all mismatched examples we have studied, a γ satisfying step 3 has been found.

In performing step 1, we favor the optimal control approach because of the inherent robustness of the resulting control obtained for the nominal system. Usually, a smaller γ is required in step 3 with a K_0 from an optimal control problem than with a K_0 from an eigenvalue placement problem. The smaller γ leads to smaller gains in the control from Eq. (4). In fact, for some problems, the ratio of the γ's for the two procedures can be on the order of 10^5. Further support of the optimal control approach in step 1 is that for mismatched systems it can lead to a robust control of the form of Eq. (4) while a repeat of steps 1–3 with K_0 obtained from eigenvalue placement may lead to the conclusion that there is no reasonable value of γ associated with this K_0 for which step 3 can be satisfied.

In the design procedure, step 3 requires the computation of

$$\lambda^* \triangleq \max_{r \in \mathscr{R}, s \in \mathscr{S}} \bar{\lambda}[L(r,s)]$$

where $\bar{\lambda}[L(r, s)]$ denotes the eigenvalue of $L(r, s)$ with the largest real part. To compute λ^*, the complex method is used.[7] This method, which is a modification of the unconstrained polytope optimization technique,[8] is a direct search method, which does not require the calculation of the derivative of $\bar{\lambda}[L(r, s)]$. The QR algorithm[9] is used to compute the eigenvalues. The full details of our numerical scheme are given in Ref. 10.

We begin the numerical procedure by guessing γ and computing λ^*. If the real part of λ^*, $\text{Re}(\lambda^*)$, is positive, γ is increased and a new λ^* calculated. This procedure is repeated until a γ with $\text{Re}(\lambda^*) < 0$ is obtained. For matched systems, we are assured that increasing γ will eventually lead to a value for which the condition in step 3 is satisfied. For mismatched systems, this convergence is not guaranteed; nevertheless, the procedure has led to robust controllers for several mismatched systems.

Based on some results for matched systems,[5] an expression can be developed for a scalar $\bar{\gamma}$ having the property that, for all $\gamma > \bar{\gamma}$, the corresponding control given by Eq. (4) is a robust stabilizing control. However, this value of $\bar{\gamma}$ is very conservative; there may be values of γ considerably less than $\bar{\gamma}$ for which the corresponding control is robust. Furthermore, the calculation of $\bar{\gamma}$ is quite complicated. Thus we prefer the procedure given previously.

Constant Uncertainty

If the uncertainty is constant, then with the control from Eq. (4), the closed-loop system is

$$\dot{x}(t) = [A_0 + \Delta A(r) + (B_0 + \Delta B(s))(K_0 - \gamma B_0'P)]x(t) \triangleq A_c(r, s)x(t) \tag{5}$$

Since $A_c(r, s)$ is independent of time, there is no need to use Lyapunov theory; eigenvalue analysis can be used. Step 3 of the design method becomes step 3′.

3′) Choose a scalar $\gamma > 0$ such that, for all $r \in \mathscr{R}$ and $s \in \mathscr{S}$, all of the eigenvalues of $A_c(r, s)$ have negative real parts.

If γ is determined so that the condition in step 3′ is satisfied, then

$$\lambda_c^* \triangleq \max_{r \in \mathscr{R}, s \in \mathscr{S}} \bar{\lambda}[A_c(r, s)]$$

has a negative real part and every solution of Eq. (5) asymptotically approaches zero for every $r \in \mathscr{R}$, $s \in \mathscr{S}$. Thus, the corresponding control is a linear robust stabilizing control.

The calculation of λ_c^* is done in the same manner as the calculation of λ^*. Once again, for matched systems, we are guaranteed the existence of a γ satisfying step 3′. For mismatched systems, no such guarantee of existence is available.

In the next section, several examples are presented.

Examples

Example 1

A version of the pitch-axis model for the AFTI/F-16 flying at 3000 ft and Mach 0.6 is considered.[11] The equations of motion are described by Eq. (1) with

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{matrix} \text{—pitch attitude (deg)} \\ \text{—pitch rate (deg/s)} \\ \text{—angle of attack (deg)} \end{matrix}$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \begin{matrix} \text{—elevator deflection (deg)} \\ \text{—flaperon deflection (deg)} \end{matrix}$$

$$A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.87 & 43.22 \\ 0 & 0.99 & -1.34 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 & 0 \\ -17.25 & -1.58 \\ -0.17 & -0.25 \end{bmatrix}$$

$$\Delta A(r) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & r_1 & r_2 \\ 0 & r_3 & r_4 \end{bmatrix}; \quad \Delta B(s) = 0$$

This system satisfies the matching assumption. We only consider uncertainty in the matrix $\Delta A(r)$; i.e., we set $\Delta B = 0$. A similar analysis could be carried out with $\Delta B \neq 0$. For the uncertainty constraint set $\mathscr{R}$, we use a set parametrized by a scalar β

$$\mathcal{R} = \{r: |r_1| \le 0.1\beta, |r_2| \le 5\beta, |r_3| \le 0.1\beta, |r_4| \le 0.15\beta\}$$

Here, $\beta = 1$ corresponds to roughly 10% error in the uncertain entries in A, $\beta = 2$–20% error, etc.

The eigenvalues of the nominal system should be at $\{-5.6 \pm 4.2j, -1\}$.[11] This placement is achieved with

$$K_0 = \begin{bmatrix} 0.9582 & 0.4167 & 3.0663 \\ 1.5331 & 0.6667 & 4.9061 \end{bmatrix}$$

The matrix K_0 was obtained using a pole-placement algorithm.[12] With this K_0, the solution of the Lyapunov equation (3) is

$$P = \begin{bmatrix} 1.1668 & 0.0244 & 0.0672 \\ 0.0244 & 0.0538 & -0.0455 \\ 0.0672 & -0.0455 & 0.4187 \end{bmatrix}$$

and the control from Eq. (4) is

$$u = \begin{bmatrix} 0.9582 + 0.4323\gamma & 0.4167 + 0.92038\gamma & 3.0663 - 0.7137\gamma \\ 1.5331 + 0.0553\gamma & 0.6667 + 0.07368\gamma & 4.9061 + 0.0328\gamma \end{bmatrix} x$$

The Lyapunov equation was solved using the algorithm given in Ref. 13. The matrix $L(r)$ in step 3 is

$$L(r) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + r_1 \begin{bmatrix} 0 & 0.0244 & 0 \\ 0.0244 & 0.1076 & -0.0455 \\ 0 & -0.0455 & 0 \end{bmatrix}$$

$$+ r_2 \begin{bmatrix} 0 & 0 & 0.0244 \\ 0 & 0 & 0.0583 \\ 0.0244 & 0.0538 & -0.091 \end{bmatrix}$$

$$+ r_3 \begin{bmatrix} 0 & 0.0672 & 0 \\ 0.0672 & -0.091 & 0.4187 \\ 0 & 0.4187 & 0 \end{bmatrix}$$

$$+ r_4 \begin{bmatrix} 0 & 0 & 0.0672 \\ 0 & 0 & -0.0455 \\ 0.0672 & -0.0455 & 0.8374 \end{bmatrix}$$

$$+ \gamma \begin{bmatrix} -0.3798 & -0.8038 & 0.6134 \\ -0.8038 & -1.7048 & 1.3088 \\ 0.6134 & 1.3088 & -1.0208 \end{bmatrix}$$

The problem becomes that of determining γ so that $L(r)$ is negative definite for all $r \in \mathcal{R}$. Since the system is matched, we know such a γ exists.

Table 1 Results for example 1

β	Time-varying uncertainty γ	Constant uncertainty γ
1	0	0
2	1	0
5	5	0
7	15	0
10	47	1

For the constant uncertainty case, we need to determine γ so that the matrix $A_c(r)$, given by

$$A_c(r) = \begin{bmatrix} 0 & 1 & 0 \\ -18.9511 & -9.1122 & -17.4257 \\ -0.5462 & 0.7525 & -3.0878 \end{bmatrix}$$

$$+ \gamma \begin{bmatrix} 0 & 0 & 0 \\ -7.5446 & -15.9915 & 12.2595 \\ -0.0873 & -0.1749 & 0.1131 \end{bmatrix}$$

$$+ r_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + r_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ r_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + r_4 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is negative definite for all $r \in \mathcal{R}$.

The results of our computations are shown in Table 1 for various values of the parameter β. The value of γ is the smallest integer for which the appropriate negative-definite condition is satisfied. As expected, for each β, the value of γ required in the time-varying case is greater than or equal to the γ needed if the uncertainty is constant.

Observe that the design based on the nominal system ($\gamma = 0$) is quite robust in the case of constant uncertainty. It can tolerate roughly 70% deviation in the uncertain coefficients and still maintain stability. However, for time-varying disturbances, the nominal control can accommodate disturbances of roughly 10% only; to guarantee stability against time-varying disturbances of 70% requires a γ of 15.

Example 2

This example is the longitudinal dynamics of the aircraft A4D at flight condition of Mach 0.9 and 15,000 ft altitude.[14,15] The state and control are

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \begin{matrix} \text{– forward velocity (ft/s)} \\ \text{– angle of attack (rad)} \\ \text{– pitch rate (rad/s)} \\ \text{– pitch angle (rad)} \end{matrix}$$

$$u = [u_1] \text{ – elevator deflection (deg)}$$

and the nominal system matrices are

$$A_0 = \begin{bmatrix} -0.061 & -32.4 & 0 & 32.2 \\ -0.0001 & -1.5 & 1 & 0 \\ -0.11 & -38.8 & -2.7 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ -0.1 \\ -33.8 \\ 0 \end{bmatrix}$$

The (3, 2) entry of A_0 is the most sensitive to parameter changes,[15] so we take

$$\Delta A(r) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{R} = \{r: |r| \le \bar{r}\}$$

In step 1, the matrix K_0 is obtained by solving the linear-quadratic optimal control problem for the nominal system with cost function

$$J = \int_0^\infty (x'Qx + u'Ru)\,dt$$

The weighting matrices Q and R are chosen by the designer. Here we take $Q = I$ and $R = 1$. If the resulting time response is not satisfactory, these matrices can be adjusted.

The linear-quadratic optimal control problem was solved using the iterative technique in Ref. 16. The details of the program are discussed in Ref. 13. This results in the nominal gain matrix

$$K_0 = [0.9583 \quad -14.096 \quad 1.1112 \quad 19.6841]$$

with eigenvalues for the nominal system at $\{-1.93 \pm 2.87j, -3.86, -32.69\}$. The solution of the Lyapunov equation (3) for this K_0 is

$$P = \begin{bmatrix} 0.5862 & -4.6055 & 0.0279 & 5.7267 \\ -4.6055 & 48.9223 & -0.332 & -63.2628 \\ 0.0279 & -0.332 & 0.0166 & 0.4651 \\ 5.7267 & -63.2628 & 0.4651 & 84.4787 \end{bmatrix}$$

which leads to

$$L(r) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + r\begin{bmatrix} 0 & 0.0279 & 0 & 0 \\ 0.0279 & -0.664 & 0.0166 & 0.4651 \\ 0 & 0.0166 & 0 & 0 \\ 0 & 0.4651 & 0 & 0 \end{bmatrix}$$

$$+ \gamma\begin{bmatrix} -0.4639 & 6.0963 & -0.5098 & -9.0477 \\ 6.0963 & -80.1175 & 6.7001 & 118.9054 \\ -0.5098 & 6.7001 & -0.5603 & -9.9439 \\ -9.0477 & 118.9054 & -9.9439 & -176.4718 \end{bmatrix}$$

in step 3 and, for constant disturbances,

$$A_c(r) = \begin{bmatrix} -0.061 & -32.4 & 0 & 32.2 \\ -0.0959 & -0.0904 & 0.8889 & -1.9684 \\ -32.5005 & 437.6448 & -40.2586 & -665.3226 \\ 0 & 0 & 1 & 0 \end{bmatrix} + r\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+ \gamma\begin{bmatrix} 0 & 0 & 0 & 0 \\ -0.0482 & 0.6329 & -0.0529 & -0.9393 \\ -16.2781 & 213.9269 & -17.8903 & -317.4969 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

in step 3′.

The results for the control

$$\begin{aligned} u(t) &= (K_0 - \gamma B_0' P)x(t) \\ &= (0.9583 + 0.4816\gamma)x_1 + (-14.096 - 6.3292\gamma)x_2 \\ &\quad + (1.1112 + 0.5293\gamma)x_3 + (19.6841 + 9.3934\gamma)x_4 \end{aligned} \tag{6}$$

are given in Table 2 for time-varying and constant disturbances. For each case, the maximum value of $\bar{r}$ is given for several values of γ.

There are several interesting observations that can be made from Table 2. For a given value of $\bar{r}_{max}$, a larger γ is required to guarantee stability for time-varying disturbances than for constant ones. Also, increasing γ beyond a certain point does not lead to a larger $\bar{r}_{max}$. This occurs because the system is not matched. For matched systems, increasing γ will lead to larger $\bar{r}_{max}$.

Example 3

The motion of a helicopter in the vertical plane has been treated by several authors.[17–20] The dynamics are given by Eq. (1) with four state variables—x_1 = horizontal velocity (knot/s), x_2 = vertical velocity (knot/s), x_3 = pitch rate (deg/s), and x_4 = pitch angle (deg)—and two control variables—u_1 = collective pitch control and u_2 = longitudinal cyclic pitch control. In the airspeed range of 60–170 knots, significant changes occur only in elements a_{32}, a_{34}, and b_{21}. For this range of operating conditions,

$$A_0 = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 \\ 0.1002 & 0.2855 & -0.707 & 1.3229 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Delta A(r) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & r_1 & 0 & r_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.0447 & -7.5922 \\ -5.52 & 4.99 \\ 0 & 0 \end{bmatrix}, \quad \Delta B(s) = \begin{bmatrix} 0 & 0 \\ s & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

with $\mathscr{R} = \{(r_1, r_2)\colon |r_1| \le 0.2192, |r_2| \le 1.2031\}$, and $\mathscr{S} = \{s\colon |s| \le 2.0673\}$. Only Yedavalli and Liang[20] obtain a robust linear control law; however, the technique used involves the determination of a state transformation and, in general, a computer search for the required transformation is needed. Here we apply the straightforward method of steps 1–4 to obtain a robust linear control for the entire range of operating conditions.

Table 2 Results for example 2

γ	Time-varying disturbances $\bar{r}_{max}$	Constant disturbances $\bar{r}_{max}$
0	1.1	200
1	20	320
5	29	460
10	31	460
20	33	460
50	34	460

Using the optimal control approach in step 1, the control

$$u(t) = \left\{ \begin{bmatrix} -0.9247 & 0.0429 & 0.9364 & 1.37764 \\ 0.0433 & 0.8387 & -0.229 & -0.7567 \end{bmatrix} - \gamma \begin{bmatrix} 0.4835 & -0.0289 & -0.4534 & -0.63377 \\ 0.0061 & -0.4883 & 0.048 & 0.1364 \end{bmatrix} \right\} x(t) \qquad (7)$$

is obtained from Eq. (4). The L matrix in step 3 with $\gamma = 1$ is

$$L(r, s) = \begin{bmatrix} -1.4677 & 0.034 & 0.4378 & 0.615 \\ 0.034 & -1.4786 & 0.0176 & 0.0964 \\ 0.4378 & 0.0176 & -1.4152 & -0.59061 \\ 0.615 & 0.0964 & -0.5904 & -1.8506 \end{bmatrix}$$

$$+ r_1 \begin{bmatrix} 0 & 0.1547 & 0 & 0 \\ 0.1547 & 0.1592 & 0.1385 & 0.0422 \\ 0 & 0.1385 & 0 & 0 \\ 0 & 0.0422 & 0 & 0 \end{bmatrix}$$

$$+ r_2 \begin{bmatrix} 0 & 0 & 0 & 0.1547 \\ 0 & 0 & 0 & 0.0796 \\ 0 & 0 & 0 & 0.1385 \\ 0.1547 & 0.0796 & 0.1385 & 0.0844 \end{bmatrix}$$

$$+ s \begin{bmatrix} -0.3901 & -0.1514 & 0.08413 & 0.2963 \\ -0.1514 & 0.0165 & 0.1681 & 0.2301 \\ 0.08413 & 0.1681 & 0.2256 & 0.1431 \\ 0.2963 & 0.2301 & 0.1431 & -0.0492 \end{bmatrix}$$

Since this matrix is negative definite for all $r \in \mathscr{R}$ and $s \in \mathscr{S}$, we conclude that the control equation (7) with $\gamma = 1$ is a linear robust tracking control. We also tried $\gamma = 0$ in step 3, but then $L(r, s)$ does not satisfy the negative-definite requirement. For the constant disturbance case, the matrix $A_c(r, s)$ in step 3′ is negative definite for $\gamma = 0$. Thus the control equation (7) with $\gamma = 0$ is a robust control for constant disturbances, but not time-varying disturbances.

Conclusions

We have considered the problem of stabilizing a linear system with time-varying uncertainty. A method for determining a linear feedback control law that stabilizes the system has been developed. This control law guarantees that the origin of the system is asymptotically stable for all admissible time variations of the uncertainty. The technique is illustrated by applying the results to three aircraft problems. The numerical aspects of determining the control law are also discussed.

Acknowledgments

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